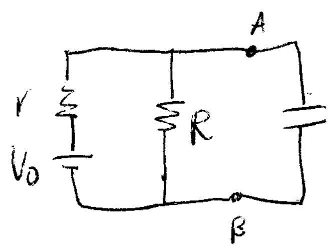
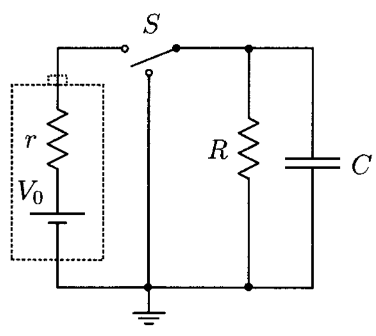
PHYSICS 231 | Final Examination | Name: Jake Bobowski
2013/14 Term 1 | | Student Number: Sol'ns

Free Response: Write out complete answers to the following questions. Show your work.

(15^{pts}) **1.** The circuit below has a battery, a switch S, and a resistor R in parallel with capacitor C. The battery is modelled as a voltage source V_0 in series with resistor r which represents the internal resistance of the battery.

w/ switch in up position



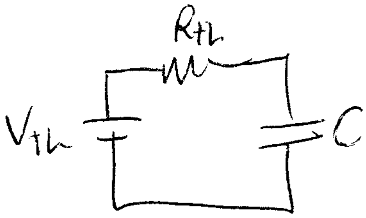


(a) Suppose that the switch S is initially in the down position for a long time and then, at time $t = 0$, it is moved to the up position. If $V_0 = 9.0$ V, $r = 1.5\ \Omega$, $R = 16$ kΩ, and $C = 22\ \mu$F, how long does it take the capacitor acquire a charge of 0.11 mC? (5 marks)

replace everything to left of terminals A & B w/ Thevenin equiv. → voltage divider

$$V_{th} = V_0 \frac{R}{r+R} \quad (\approx V_0)$$

$$R_{th} = \frac{rR}{r+R} \quad (\approx r)$$



$$V_c(t) = V_{th}\left(1 - e^{-t/\tau}\right) \qquad \tau = R_{th} C$$

$$\therefore \frac{V_c(t)}{V_{th}} = 1 - e^{-t/\tau} \qquad -\frac{t}{\tau} = \ln\left(1 - \frac{V_c(t)}{V_{th}}\right)$$

$$t = -\tau \ln\left(1 - \frac{V_c(t)}{V_{th}}\right)$$

$$C = \frac{q}{V_c} \quad \therefore V_c(t) = \frac{q(t)}{C}$$

$$\therefore t = -\tau \ln\left(1 - \frac{q/C}{V_{th}}\right) \qquad \tau = R_{th} C = \frac{rR}{r+R} C = 33.0\ \mu s$$

$$V_{th} = 9.00\ \text{V}$$

$$\boxed{\therefore t = 26.8\ \mu s}$$

15 pts

(b) Now suppose that the switch has been in the up position for an unknown amount of time. Then, at $t = 0$, it is moved into the down position to discharge the capacitor. The voltage across the capacitor V_c and its uncertainty are measured as a function of time as shown in the table below. Also shown in the table are the values of $\ln V_c$. Complete the table by entering the error in $\ln V_c$ in the final column of the table. Clearly explain how these values are obtained. (5 marks)

t (s)	V_c (V)	ΔV_c (V)	$\ln V_c$	$\Delta(\ln V_c)$
0.1	5.4	0.2	1.68	0.04
0.2	3.9	0.2	1.36	0.05
0.3	3.1	0.2	1.13	0.06
0.6	1.31	0.1	0.27	0.08
0.9	0.55	0.05	−0.60	0.09

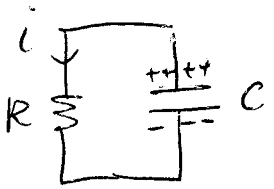
$$\Delta(\ln V_c) = \left| \frac{\partial \ln V_c}{\partial V_c} \Delta V_c \right|$$

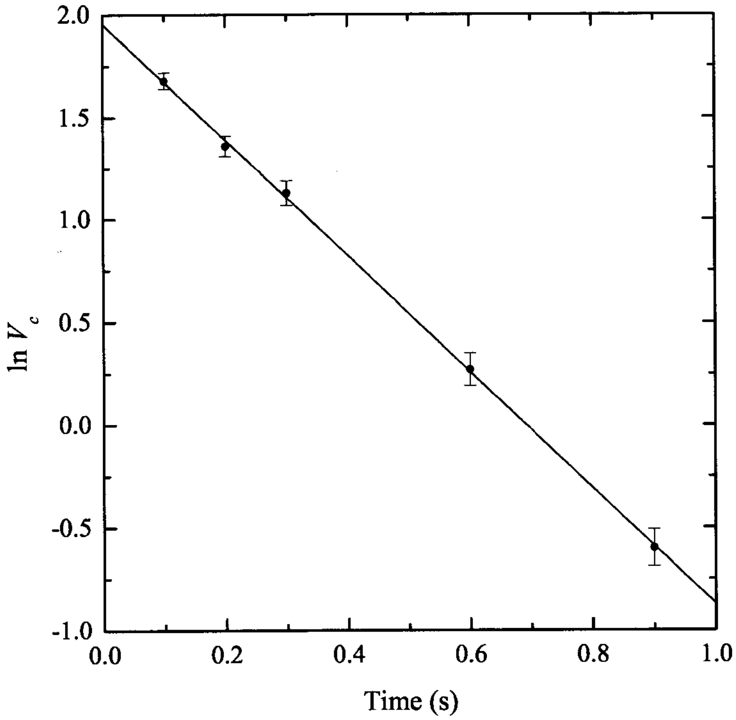
$$\therefore \boxed{\Delta(\ln V_c) = \frac{\Delta V_c}{V_c}}$$

0 pts

(c) The $\ln V_c$ versus t data from the table are plotted below and fit to a straight line. The equation of the best-fit line is $y = mx + b$ with $m = -2.83 \pm 0.05\ \mathrm{s}^{-1}$ and $b = 1.95 \pm 0.02$. Use this information to determine the initial voltage across the capacitor at $t = 0$ and its uncertainty $[V_c(0) \pm \Delta V_c(0)]$ and the time constant of the decay and its uncertainty $[\tau \pm \Delta\tau]$. Does the experimentally determined value of τ agree with the expected value? (5 marks)

w/ switch in down position circuit becomes





For discharging cap $\quad V_c(t) = V_c(0)\, e^{-\frac{t}{\tau}}$ $\quad$ where $\tau = RC$

$= (16\,k\Omega)(22\,\mu F)$

$\therefore \ln V_c(t) = \ln V_c(0) - \dfrac{t}{\tau}$ $\quad = 0.352\ s$

eq'n of straight line

w/ $b = \ln V_c(0)$ $\quad m = -\dfrac{1}{\tau}$

experimental value agrees w/ expected value.

$\therefore V_c(0) = e^{b}$

$\Delta V_c(0) = \left| \dfrac{\partial e^{b}}{\partial b} \Delta b \right| = e^{b} \Delta b$

$\boxed{\therefore V_c(0) = 7.03 \pm 0.14\ V}$

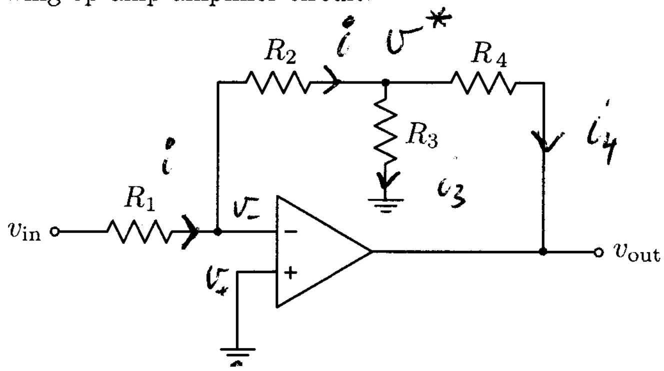
$\tau = -\dfrac{1}{m}$

$\Delta\tau = \left| \dfrac{\partial(-\frac{1}{m})}{\partial m} \Delta m \right| = \left| \dfrac{\Delta m}{m^2} \right|$

$\boxed{\therefore \tau = 0.353 \pm 0.006\ s}$

0 pts

(10pts) **2.** Consider the following op-amp amplifier circuit:



(a) Find an expression for the gain ($G = v_{out}/v_{in}$) of the amplifier in terms of R_1, R_2, R_3, and R_4. (7 marks)

op amp golden rules: $i_- = i_+ = 0$

$v_+ = v_- = 0$

$v_{in} - iR_1 = v_- = 0$ $\quad \boxed{\therefore i = \dfrac{v_{in}}{R_1}}$

$\overset{0}{\cancel{v_-}} - iR_2 = v^*$

$\therefore v^* = -iR_2 = -v_{in}\dfrac{R_2}{R_1}$

$v^* - i_3R_3 = 0$

$\therefore i_3 = \dfrac{v^*}{R_3} = \boxed{-v_{in}\dfrac{R_2}{R_1R_3}}$

$v^* - i_4R_4 = v_{out}$

$\therefore i_4 = \dfrac{v^* - v_{out}}{R_4} = \boxed{-v_{in}\dfrac{R_2}{R_1R_4} - \dfrac{v_{out}}{R_4}}$

By junction rule

$i = i_3 + i_4$

$\therefore \dfrac{v_{in}}{R_1} = -v_{in}\dfrac{R_2}{R_1R_3} - v_{in}\dfrac{R_2}{R_1R_4} - \dfrac{v_{out}}{R_4}$

solve for v_{out}

$\dfrac{v_{out}}{R_4} = -v_{in}\left[\dfrac{R_2}{R_1R_3} + \dfrac{R_2}{R_1R_4} + \dfrac{1}{R_1}\right]$

$\therefore v_{out} = -v_{in}\left[\dfrac{R_2R_4}{R_1R_3} + \dfrac{R_2}{R_1} + \dfrac{R_4}{R_1}\right]$

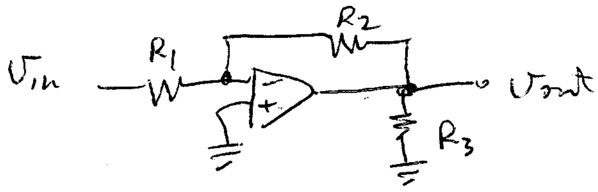
$\boxed{= -\dfrac{v_{in}}{R_1R_3}\left[R_2R_4 + R_2R_3 + R_3R_4\right]}$

10 pts

(b) What does the gain become if $R_4 = 0$? Does this result agree with what you would expect? Explain. (2 marks)

If $R_4 = 0$ $\quad V_{out} = -\dfrac{V_{in}}{R_1 R_3}\left[R_2 R_3\right] = -V_{in}\dfrac{R_2}{R_1}$

This is the inverting amplifier result.
w/ $R_4 = 0$ circuit becomes



Here R_3 is irrelevant & have inverting amp circuit.

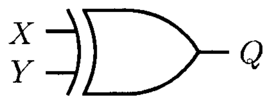
(c) What is the gain of the amplifier if $R_1 = R_3 = 1\ \text{k}\Omega$ and $R_2 = R_4 = 100\ \text{k}\Omega$? (1 mark)

$$\frac{V_{out}}{V_{in}} = -\frac{1}{(1\,k\Omega)(1\,k\Omega)}\left[(100\,k\Omega)(100\,k\Omega) + (100\,k\Omega)(1\,k\Omega) + (1\,k\Omega)(100\,k\Omega)\right]$$

$$= -10^4 + 2\cdot 10^2 = \boxed{-10,200}$$

0 pts

(10pts) **3.** The XOR gate and its truth table are shown below.

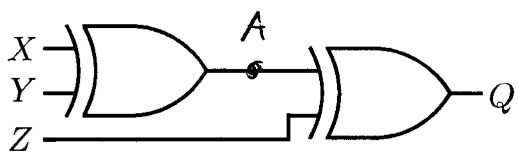


X	Y	$Q = X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

The XOR gate can be thought of as a two-input "parity" checker. If an odd number of inputs are high, then the output is high. Alternatively, if zero or an even number of the inputs are high, then the output is low.

(a) Confirm that the circuit below acts as a three-input parity checker. Do this by writing out the complete truth table for this circuit and confirming that it works as required. (4 marks)

3 inputs, 2^3 possible input combos ~ 8



	X	Y	Z	$A = X \oplus Y$	$Q = A \oplus Z$	
zero	0	0	0	0	0	✓
odd	0	0	1	0	1	✓
odd	0	1	0	1	1	✓
even	0	1	1	1	0	✓
odd	1	0	0	1	1	✓
even	1	0	1	1	0	✓
even	1	1	0	0	0	✓
odd	1	1	1	0	1	✓

works as expected

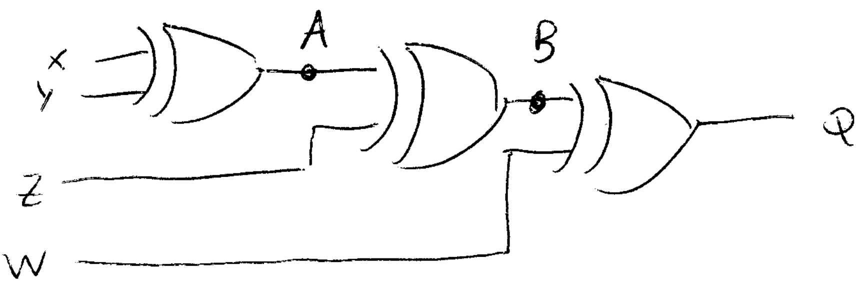
→ 3 input parity checker

10 pts

(6) Design a four-input parity checking circuit. Construct the complete truth table to confirm that the circuit that you designed works as required. (6 marks)

Try

$2^4 \rightarrow 16$ possible input combos



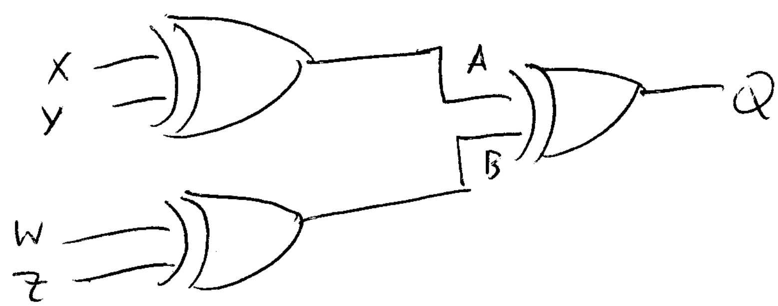
X	Y	Z	W	$A = X \oplus Y$	$B = A \oplus Z$	$Q = B \oplus W$	even or odd?
0	0	0	0	0	0	0	zero ✓
0	0	0	1	0	0	1	odd ✓
0	0	1	0	0	1	1	odd ✓
0	0	1	1	0	1	0	even ✓
0	1	0	0	1	1	1	odd ✓
0	1	0	1	1	1	0	even ✓
0	1	1	0	1	0	0	even ✓
0	1	1	1	1	0	1	odd ✓
1	0	0	0	1	1	1	odd ✓
1	0	0	1	1	1	0	even ✓
1	0	1	0	1	0	0	even ✓
1	0	1	1	1	0	1	odd ✓
1	1	0	0	0	0	0	even ✓
1	1	0	1	0	0	1	odd ✓
1	1	1	0	0	1	1	odd ✓
1	1	1	1	0	1	0	even ✓

works as required

4-input parity checker.

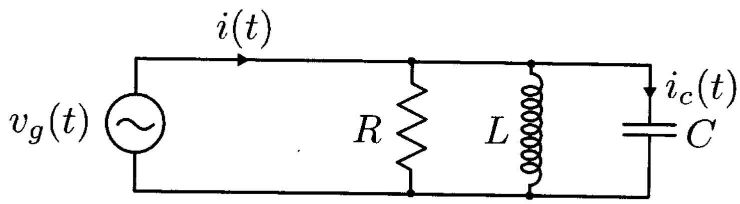
0 pts

Note that the circuit below also works



X Y W Z	$A = X \oplus Y$	$B = W \oplus Z$	$Q = A \oplus B$	even or odd
0 0 0 0	0	0	0	zero
0 0 0 1	0	1	1	odd
0 0 1 0	0	1	1	odd
0 0 1 1	0	0	0	even
0 1 0 0	1	0	1	odd
0 1 0 1	1	1	0	even
0 1 1 0	1	1	0	even
0 1 1 1	1	0	1	odd
1 0 0 0	1	0	1	odd
1 0 0 1	1	1	0	even
1 0 1 0	1	1	0	even
1 0 1 1	1	0	1	odd
1 1 0 0	0	0	0	even
1 1 0 1	0	1	1	odd
1 1 1 0	0	1	1	odd
1 1 1 1	0	0	0	even

(20[pts]) **4.** Consider the parallel LRC circuit shown below. The function generator supplies $v_g(t) = V_0 \sin \omega t$ to the circuit.



(a) Find expressions for the amplitude I_0 and phase ϕ of the current $i(t) = I_0 \sin(\omega + \phi)$ shown in the figure above. Give your expressions in terms of R, L, C, ω, and V_0. (8 marks)

$$I_0 = \frac{V_0}{|Z|} \qquad \tan\phi = \frac{\mathrm{Im}[Z^*]}{\mathrm{Re}[Z]}$$

$$Z = R \,||\, Z_L \,||\, Z_C$$

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$\therefore \frac{1}{|Z|^2} = \left[\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)\right]\left[\frac{1}{R} - j\left(\omega C - \frac{1}{\omega L}\right)\right] = \frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2$$

$$\therefore \frac{1}{|Z|} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

$$\boxed{\therefore I_0 = V_0 \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}}$$

$$Z = \frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)} \cdot \frac{\frac{1}{R} - j\left(\omega C - \frac{1}{\omega L}\right)}{\frac{1}{R} - j\left(\omega C - \frac{1}{\omega L}\right)} = \frac{\frac{1}{R} - j\left(\omega C - \frac{1}{\omega L}\right)}{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

$$\therefore Z^* = \frac{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)}{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

$$\therefore \tan\phi = \frac{+\left(\omega C - \frac{1}{\omega L}\right)}{\cancel{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} \cdot \frac{\cancel{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}}{\frac{1}{R}}$$

$$\boxed{\therefore \tan\phi = R\left(\frac{-1}{\omega L} + \omega C\right)}$$

20 pts

(b) For values of $\omega \gg 1/\sqrt{LC}$, show that I_0 can be approximated as $I_0 \approx A\omega^n$. Determine A and n. (2 marks)

for $\omega \to \infty$ $\quad I_0 \approx V_0\sqrt{\frac{1}{R^2}+(\omega C)^2} \approx V_0 C\omega$ $\quad \boxed{\therefore A = V_0 C \quad n = 1}$

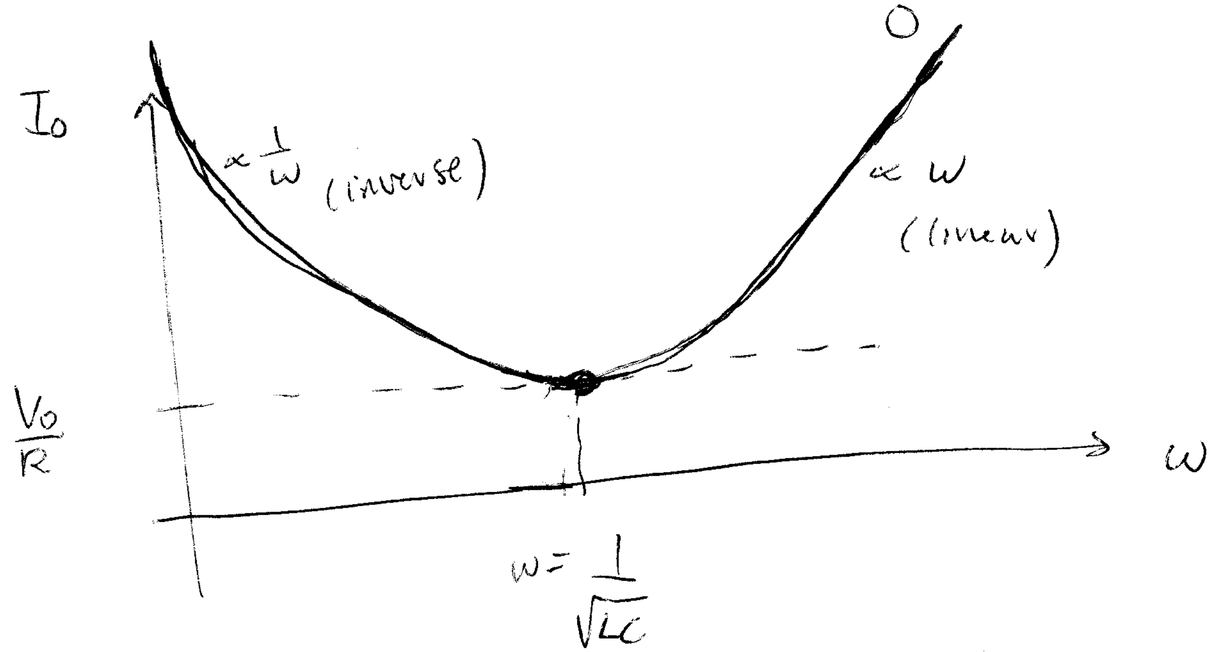
(c) For values of $\omega \ll 1/\sqrt{LC}$, show that I_0 can be approximated as $I_0 \approx B\omega^m$. Determine B and m. (2 marks)

for $\omega \to 0$ $\quad I_0 \approx V_0\sqrt{\frac{1}{R^2}+\left(-\frac{1}{\omega L}\right)^2} \approx \frac{V_0}{\omega L}$ $\quad \boxed{\therefore B = \frac{V_0}{L} \quad m = -1}$

(c) Sketch a plot ~~plots~~ of I_0 versus ω ~~and ϕ versus ω~~. On your plot~~s~~, label the point~~s~~ where $\omega = 1/\sqrt{LC}$. (4 marks)

when $\omega = \frac{1}{\sqrt{LC}}$ $\quad I_0 = V_0\sqrt{\frac{1}{R^2}+\left(\frac{C}{\sqrt{LC}} - \frac{\sqrt{LC}}{L}\right)^2}$

$$= V_0\sqrt{\frac{1}{R^2}+\Big(\underbrace{\sqrt{\tfrac{C}{L}} - \sqrt{\tfrac{C}{L}}}_{0}\Big)^2} = \frac{V_0}{R}$$



0 pts

(10pts) **5.** Make use of Euler's equation $\left(e^{\pm j\phi} = \cos\phi \pm j\sin\phi\right)$ to help you evaluate the following integral:

$$\int_0^{\pi/4} e^x \cos x \, dx$$

You must show all of your work to receive full credit.

$$\left.\begin{aligned} e^{jx} &= \cos x + j\sin x \\ e^{-jx} &= \cos x - j\sin x \end{aligned}\right\} \text{add these two}$$

$$e^{jx} + e^{-jx} = 2\cos x \qquad \therefore \cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

difference gives $\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$

$$\therefore \int_0^{\pi/4} e^x \cos x \, dx = \int_0^{\pi/4} e^x \frac{\left(e^{jx} + e^{-jx}\right)}{2} dx = \frac{1}{2}\int_0^{\pi/4}\left(e^{(1+j)x} + e^{(1-j)x}\right)dx$$

$$= \frac{1}{2}\left[\frac{1}{1+j}e^{(1+j)x} + \frac{1}{1-j}e^{(1-j)x}\right]\Bigg|_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{e^x e^{jx}}{1+j} + \frac{e^x e^{-jx}}{1-j}\right]\Bigg|_0^{\pi/4} = \frac{1}{2}\left(\frac{1-j}{2}e^x e^{jx} + \frac{1+j}{2}e^x e^{-jx}\right)\Bigg|_0^{\pi/4}$$

$$= \frac{1}{4}\left[\left(e^x e^{jx} + e^x e^{-jx}\right) - j\left(e^x e^{jx} - e^x e^{-jx}\right)\right]\Bigg|_0^{\pi/4}$$

$$= \frac{1}{2}\left[e^x\left(\frac{e^{jx} + e^{-jx}}{2}\right) + e^x\left(\frac{e^{jx} - e^{-jx}}{2j}\right)\right]\Bigg|_0^{\pi/4}$$

$$= \frac{e^x}{2}\left(\cos x + \sin x\right)\Bigg|_0^{\pi/4}$$

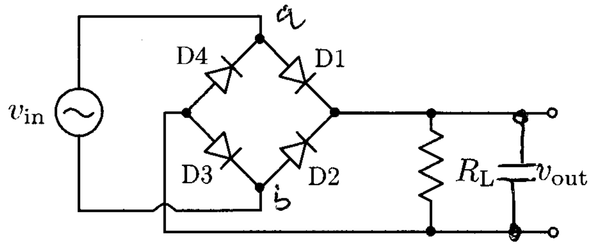
$$= \frac{e^{\pi/4}}{2}\left(\cancelto{\frac{1}{\sqrt{2}}}{\cos\pi/4} + \cancelto{\frac{1}{\sqrt{2}}}{\sin\pi/4}\right) - \cancelto{1}{\frac{e^0}{2}}\left(\cancelto{1}{\cos 0} + \cancelto{0}{\sin 0}\right)$$

10 pts

$$\therefore \int_0^{\pi/4} e^x \cos x \, dx = \frac{e^{\pi/4}}{2} \frac{2}{\sqrt{2}} - \frac{1}{2}$$

$$= \boxed{\frac{e^{\pi/4}}{\sqrt{2}} - \frac{1}{2} = 1.0509}$$

(10pts) **6.** The circuit below consists of a sinusoidal input v_{in}, four diodes, and a resistor R_{L}. The output of the circuit v_{out} is taken to be the voltage across R_{L}.



(a) The next page shows a plot of v_{in} versus time. On the axes directly below, sketch a plot of v_{out} versus time. Draw your plot to scale as best you can. (7 marks)

(b) With the addition of a single capacitor C, this circuit can be made into an AC-to-DC converter. Add the required capacitor to the circuit above. How should the product $R_{\text{L}}C$ compare to the period T of the AC input v_{in} in order for the circuit to work as an effective AC-to-DC converter? (3 marks)

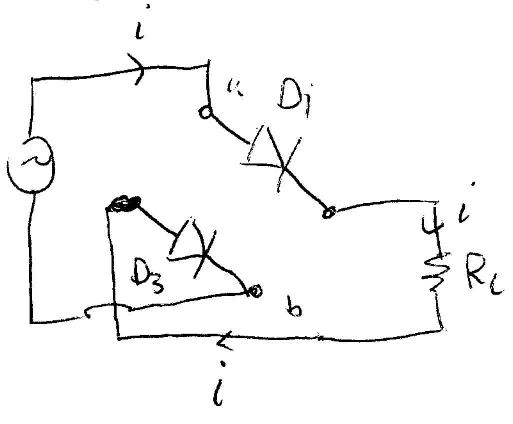
(b) Want the capacitor discharge time to be slow.

$\therefore$ require $R_L C \gg T$.

(a) assume V_{in} is in positive part of cycle. $\rightarrow$ a positive w.r.t. b

D_1 forward biased

D_3 forward biased.



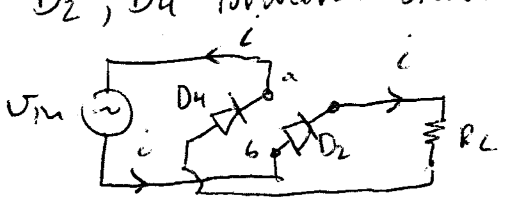
$$V_{in} - V_{D1} - \underbrace{iR_L}_{V_{out} = iR_L} - V_{D3} = 0$$

$$\therefore V_{out} = V_{in} - V_{D1} - V_{D3} \simeq V_{in} - 2V_D$$

where $V_D \simeq 0.6$ V for forward biased diode.

when V_{in} is in neg. part of cycle $\rightarrow$ b positive. w.r.t. a

D_2, D_4 forward biased.



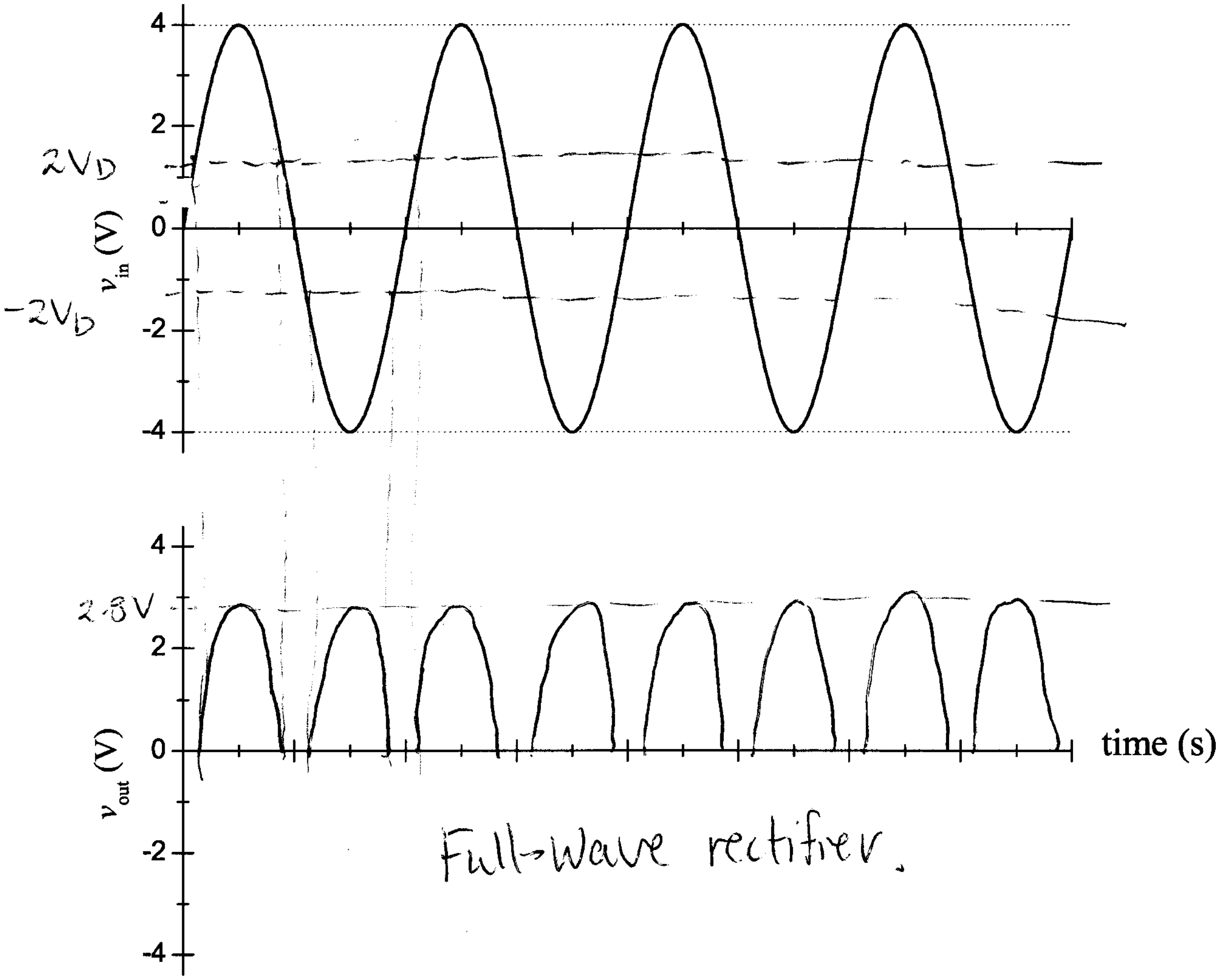
$V_{out} = +iR_L$

$-V_{in} - V_{D2} - iR_L - V_{D4} = 0$

$\therefore V_{out} = -V_{in} - 2V_D$

10 pts

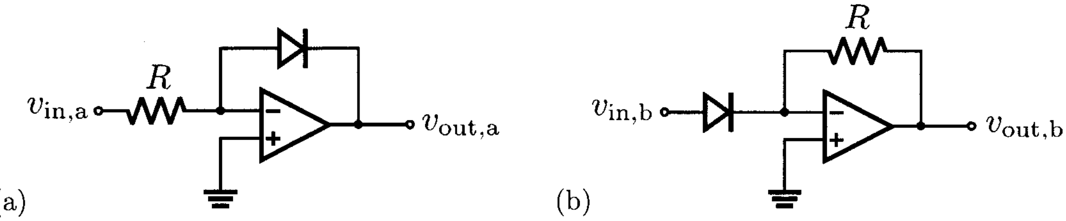
$4\,V - 2V_D \approx 4V - 1.2V = 2.8\,V$



Full-wave rectifier.

0 pts

(10pts) **7.** Figures (a) and (b) below show the log- and the anti-log-amplifiers respectfully.



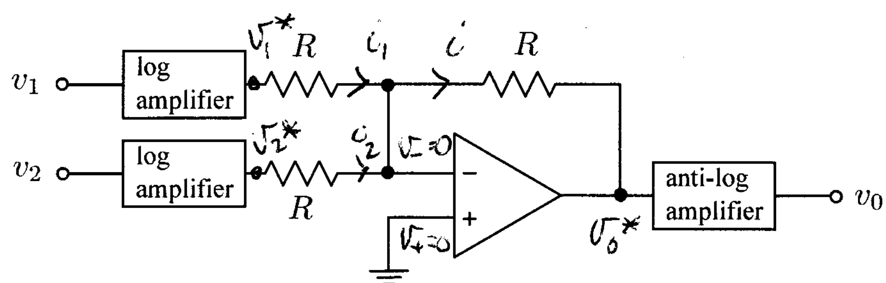
For suitable $v_{in,a}$ & $v_{in,b}$,

The outputs of the circuits are given by:

$$v_{\text{out,a}} = -\frac{k_B T}{e} \ln \frac{v_{\text{in,a}}}{I_0 R}$$

$$v_{\text{out,b}} = -I_0 R \exp\left(\frac{e\, v_{\text{in,b}}}{k_B T}\right)$$

The circuit below makes use of the log and anti-log amplifiers. In this circuit the box labelled "log amplifier" corresponds to circuit (a) above and the box labelled "anti-log amplifier" corresponds to circuit (b) above. Assume that all resistors used in this problem are identical and that all diodes used in this problem are identical.



Find an expression for v_0 of this circuit. What kind of function does this circuit perform?

$v_1^* - i_1 R = v_- = 0 \quad \therefore i_1 = \frac{v_1^*}{R}$ likewise $i_2 = \frac{v_2^*}{R}$

$i = i_1 + i_2 = \frac{v_1^* + v_2^*}{R}$

$\underbrace{v_-}_{0} - iR = v_0^* \quad \therefore v_0^* = -\left(\frac{v_1^* + v_2^*}{R}\right)R = -(v_1^* + v_2^*)$

$v_1^* = -\frac{k_B T}{e} \ln\left(\frac{v_1}{I_0 R}\right) \qquad v_2^* = -\frac{k_B T}{e} \ln\left(\frac{v_2}{I_0 R}\right)$

$\therefore v_0^* = \frac{k_B T}{e}\left[\ln \frac{v_1}{I_0 R} + \ln \frac{v_2}{I_0 R}\right] = \frac{k_B T}{e} \ln\left(\frac{v_1 v_2}{(I_0 R)^2}\right)$

10 pts

$$V_o = -I_o R \exp\left[\frac{e V_o^{*}}{k_B T}\right]$$

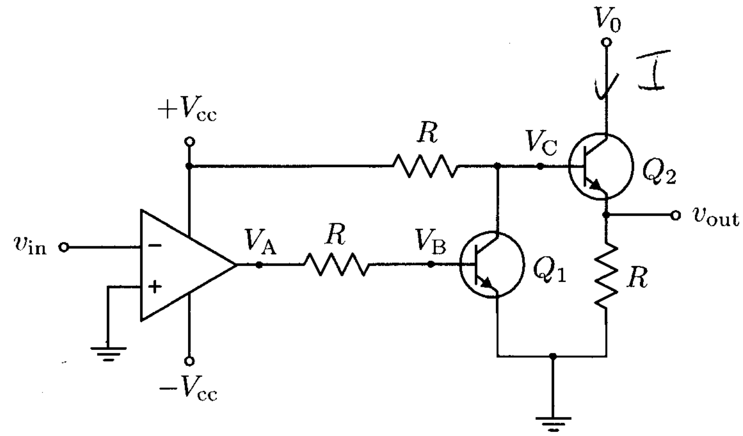
$$= -I_o R \exp\left[\frac{\cancel{e}}{\cancel{k_B T}} \frac{\cancel{k_B T}}{\cancel{e}} \ln\left(\frac{V_1 V_2}{(I_o R)^2}\right)\right]$$

$$= -I_o R \frac{V_1 V_2}{(I_o R)^2}$$

$$\boxed{\therefore V_o = -\frac{V_1 V_2}{I_o R}}$$

This circuit takes the product of V_1 & V_2.

(10pts) **8.** The circuit shown below is similar (but not identical) to a circuit that you built and studied in Experiment #6.



Assume that $R = 10$ kΩ, $V_{cc} = 15$ V, and $V_0 = 5$ V. In this problem you are simply asked to complete the table shown below. For the input voltages v_{in} shown in the table, enter the resulting values for V_A, V_B, V_C, and v_{out}. In the Q_1 and Q_2 columns indicate whether each transistor is conducting or not conducting. If a transistor is in its conducting state, enter "ON". If a transistor is not conducting, enter "OFF".

v_{in} (V)	V_A (V)	V_B (V)	V_C (V)	Q_1	Q_2	v_{out} (V)
−3	+15 V	+15 V	0	ON	OFF	0
−1	+15 V	+15 V	0	ON	OFF	0
+1	−15 V	−15 V	+15 V	OFF	ON	+5 V
+5	−15 V	−15 V	+15 V	OFF	ON	+5 V

- $V_A = A_{OL}(\cancel{v_+}^{0} - v_-) = -A_{OL} v_-$ where A_{OL} large.

  $\therefore$ if $v_- < 0$, $V_A \to +V_{cc}$

  $v_- > 0$, $V_A \to -V_{cc}$

- current into base of transistor very small. $\therefore V_A \approx V_B$
- If voltage at transistor base exceeds the emitter voltage by about 0.6 V it is conducting (ON). Otherwise it is OFF
- If Q_1 is conducting $V_C \approx 0$. If Q_1 OFF $V_C \simeq V_{cc}$
- If Q_2 conducting $v_{out} \approx IR \simeq V_0$, If Q_2 OFF $v_{out} \approx IR = 0$ since $I \approx 0$

10 pts